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Finite Element Approximation of Vector-Valued Hemivariational Problems

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Abstract. In this paper we develop a finite element approximation for vector-valued hemivariational inequalities. This class of hemivariational problems was introduced in [12],[13]. We study two different problems: unconstrained one and constrained one with a nonempty, closed, convex constraint set K.

We shall show firstly that the discrete problems are solvable by using consequences of Kakutani fixed point theorem and secondly that the solutions of the discrete problems are close on subsequences to the continuous ones.

Key words: vector-valued hemivariational inequality, finite element method, nonconvex energy function

1. Introduction

Hemivariational inequalities (HVI) introduced by Panagiotopoulos [15]–[17] can be considered as a generalization of variational inequalities. By means of them, problems with nonmonotone and multivalued constitutive laws can be formulated and solved. For the mathematical theory and the applications of (HVI) we refer to [12],[17] and the extensive bibliographies therein.

In this paper we shall present a finite element approximation of vector-valued (HVI) being a straightforward extension of the approximation of scalar-valued (HVI) presented in [7]–[10]. A similar type approximation model has been used also in [4],[5] for the elliptic variational inequalities of the second kind. But due to the nonmonotone nature of (HVI) the treatment of our problem is more involved. The outline of this paper is as follows. In the first section we shall present vector-valued (HVI) involving a nonmonotone multivalued relation in some part of a domain $\Omega \subset R^N$ (problem (P1)) or on some part of the boundary of Ω (problem (P2)), and state sufficient conditions guaranteeing their solvability. For the proofs of the existence results we refer to [12],[13]. Then we analyze a finite element approximation for the problem (P1) only, because the problem (P2) can be treated in a similar way. In the second section we show that there exists at least one solution of the approximation problem by using a consequence of Kakutani fixed point theorem presented [1]. Then we show that the solutions of the discrete problems tend on subsequences to the solutions of the continuous one. In the third section we consider the approximation of vector-valued (HVI) having a nonempty, convex, closed constraint set. For this problem we shall prove the same results as for the unconstrained one. In the last section we present an example of vector-valued (HVI), a nonmonotone skin friction in plane elasticity, and discuss how its approximation can be constructed.

2. Statement of the Problem

Let $\Omega \subset R^N$ be a bounded domain with Lipschitz boundary Γ. Let V be a real Hilbert space equipped with the norm $\|\cdot\|_V$, V' the dual space of V with the norm $\|\cdot\|_{V'}$ and $\langle\cdot,\cdot\rangle_V$ the corresponding duality pairing. We shall divide our considerations into two different cases. Firstly we shall study (HVI) having a nonmonotone multivalued condition on a subdomain Ω_0 of Ω and then (HVI) given by a nonmonotone multivalued condition on some set Γ_0 open in Γ. In applications in mechanics of solids the Hilbert space V is typically a subspace of $H^1(\Omega; R^M)$. Finally let us denote $Y_1 = L^2(\Omega_0; R^M)$ and $Y_2 = L^2(\Gamma_0; R^M)$. We shall identify Y_1, Y_2 with their dual spaces Y_1', Y_2', respectively. In Y_1 and Y_2 we use as the duality pairings the standard L^2-inner products, i.e.

$$\langle y, z\rangle_{Y_1} = \int_{\Omega_0} y(x)\cdot z(x)\,\mathrm{d}x = \int_{\Omega_0}\sum_{i=1}^{M} y_i(x)z_i(x)\,\mathrm{d}x,$$

$$\langle y, z\rangle_{Y_2} = \int_{\Gamma_0} y(x)\cdot z(x)\,\mathrm{d}s = \int_{\Gamma_0}\sum_{i=1}^{M} y_i(x)z_i(x)\,\mathrm{d}s,$$

where $y = (y_1, ..., y_M)$ and $z = (z_1, ..., z_M)$, and the norms are the standard L^2-norms induced by the above inner products.

For describing the nonmonotone multi-valued relations we introduce a locally Lipschitz continuous function $j : R^M \to R$. Let us first define what we mean by the generalized directional derivatives and the generalized gradients (in Clarke's sense) of a locally Lipschitz continuous function [3]:

DEFINITION 1. Let X be a Banach space and $f : X \to R$ a locally Lipschitz continuous near a point $x \in X$. The generalized directional derivative of f at x in the direction z is defined as follows:

$$f^\circ(x; z) = \limsup_{y\to x, t\to 0+} \frac{f(y+tz) - f(y)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is the subset of X' given by

$$\partial f(x) = \{y \in X' : f^\circ(x; z) \geq \langle y, z\rangle_X \quad \forall z \in X\}.$$

Let us turn back to our problem. The function j is supposed to satisfy firstly *the generalized sign condition* which is expressed by means of the generalized directional derivative of j. It reads as follows:

$$j^{\circ}(\xi; -\xi) \leq C_1 + C_2|\xi|^q \quad \forall \xi \in R^M, \tag{1}$$

where C_1 and C_2 are positive constants independent of ξ and $1 \leq q < 2$. Secondly, the function j fulfills the following *growth condition* expressed by means of the generalized gradient of j:

$$\eta \in \partial j(\xi) \Longrightarrow |\eta| \leq C_3(1 + |\xi|), \tag{2}$$

where C_3 is a positive constant independent of ξ and η. We assume that the bilinear form $a : V \times V \to R$ satisfies the standard continuity and coercivness conditions:

$$|a(v, w)| \leq m\|v\|_V \|w\|_V \quad \forall v, w \in V; \tag{3}$$

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V, \tag{4}$$

where m and α are positive constants. Let g be an element of V'.

By a vector-valued hemivariational inequality we mean the problem

$$\begin{cases} \text{find } u \in V \text{ and } \mathcal{X}(u) \in Y_1 \text{ such that} \\ a(u, v) + \int_{\Omega_0} \mathcal{X} \cdot v \,\mathrm{d}x = \langle g, v \rangle_V \quad \forall v \in V \\ \text{and } \mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega_0 \end{cases} \tag{P1}$$

or

$$\begin{cases} \text{find } u \in V \text{ and } \mathcal{X}(u) \in Y_2 \text{ such that} \\ a(u, v) + \int_{\Gamma_0} \mathcal{X} \cdot v \,\mathrm{d}s = \langle g, v \rangle_V \quad \forall v \in V \\ \text{and } \mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Gamma_0. \end{cases} \tag{P2}$$

REMARK 1. A simple example of j which satisfies (1) and (2) is a minimum function of two convex quadratic functions, i.e.,

$$j(\xi) = \min\{f_1(\xi), f_2(\xi)\} \qquad \forall \xi \in R^M.$$

Now, we see easily that j satisfies (2), because its generalized gradient $\partial j(\xi)$ at a point ξ belongs to the convex hull of $\{\nabla f_i(\xi) : i = 1, 2\}$. Moreover, since $j^0(\xi, \eta) = \max\{\xi^* \cdot \eta : \xi^* \in \partial j(\xi)\}$ and the fact that $\nabla f_1(\xi) \cdot \xi, \nabla f_2(\xi) \cdot \xi \geq 0$, we see that

$$\begin{aligned} j^0(\xi; -\xi) &\leq \max\{\xi^* \cdot (-\xi) : \xi^* = \lambda \nabla f_1(\xi) + (1 - \lambda)\nabla f_2(\xi), \lambda \in [0, 1]\} \\ &\leq 0 \end{aligned}$$

implying (1).

REMARK 2. Now we explain the relation between problem (P1) and a basic scalar hemivariational inequality introduced by Panagiotopoulos (for details see [16]). Let b be a function from R to R such that:

$$b \in L^\infty_{\text{loc}}(R); \tag{5}$$

there exists $\bar{\xi} > 0$ such that

$$\text{ess sup}_{\xi \in (-\infty, -\bar{\xi})} b(\xi) \leq 0 \leq \text{ess inf}_{\xi \in (\bar{\xi}, \infty)} b(\xi). \tag{6}$$

For any $\varepsilon > 0$ we define two auxiliary functions:

$$\underline{b}_\varepsilon(\xi) = \text{ess inf}_{|\tau - \xi| \leq \varepsilon} b(\tau), \qquad \overline{b}_\varepsilon(\xi) = \text{ess sup}_{|\tau - \xi| \leq \varepsilon} b(\tau),$$

and letting $\varepsilon \to 0+$ we get the upper and the lower bounds for a multivalued function $\hat{b} : R \to R$ as follows:

$$\underline{b}(\xi) = \lim_{\varepsilon \to 0+} \underline{b}_\varepsilon(\xi), \qquad \overline{b}(\xi) = \lim_{\varepsilon \to 0+} \overline{b}_\varepsilon(\xi)$$

giving

$$\hat{b}(\xi) = [\underline{b}(\xi), \overline{b}(\xi)].$$

The basic scalar hemivariational inequality reads as follows:

$$\begin{cases} \text{find } u \in V \text{ and } \mathcal{X}(u) \in L^1(\Omega_0) \cap V' \text{ such that} \\ a(u,v) + \langle \mathcal{X}, v \rangle_V = \langle g, v \rangle_V \quad \forall v \in V, \\ \mathcal{X}(x) \in \hat{b}(u(x)) \qquad \text{a.e. in } \Omega_0 \text{ and} \\ \langle \mathcal{X}, v \rangle = \int_{\Omega_0} \mathcal{X}(x) v(x) \, \mathrm{d}x \qquad \forall v \in V \cap L^\infty(\Omega_0). \end{cases} \tag{P1)'$$

If the locally Lipschitz continuous function $j : R \to R$ is defined by the relation

$$j(\xi) = \int_0^\xi b(\tau) \, \mathrm{d}\tau,$$

it is straightforward to see that the generalized gradient of j satisfies

$$\partial j(\xi) \subset \hat{b}(\xi), \quad \text{for any } \xi \in R. \tag{7}$$

This implies that every solution of (P1) is now a solution of (P1)', but not necessarily vice versa. We have the equality in (7), if the limits $\lim_{\xi \to \tilde{\xi}+} b(\xi)$ and $\lim_{\xi \to \tilde{\xi}-} b(\xi)$ exist for any $\tilde{\xi} \in R$. It is easy to see that the condition (2) is more restrictive than (5), but, on the contrary, the condition (6) is more restrictive than (1). Then, of course, (P1) and (P1)' are equivalent provided that (2), (6) and the equality in (7) hold. On the other hand, (P1) permits to treat the vector case, as well.

Let us transform (P1) and (P2) to equivalent operator inclusions, which read as follows:

$$\text{find } u \in V \text{ such that } 0 \in Au - g + T_1 u; \tag{P1}$$

or

$$\text{find } u \in V \text{ such that } 0 \in Au - g + T_2 u, \tag{P2}$$

where A is a linear operator from V to V' defined by $\langle Au, v\rangle_V = a(u,v)$ for all $u, v \in V$ and T_1, T_2 are the set-valued operators from V to V' defined by

$$T_1 v = \{w \in L^2(\Omega_0; R^M) : w(x) \in \partial j(v(x)) \text{ a.e . in } \Omega_0\};$$
$$T_2 v = \{w \in L^2(\Gamma_0; R^M) : w(x) \in \partial j(v(x)) \text{ a.e . in } \Gamma_0\}.$$

To prove that (P1) or (P2) has a solution one can use, e.g. the Galerkin approach presented in Chapter 5 of [12]. In this case it is natural to use the former formulation of the problems. The other possibity is to prove that $A(\cdot) - g + T_1(\cdot)$ or $A(\cdot) - g + T_2(\cdot)$ are pseudo-monotone operators and to apply the abstract results for that type of operators. This approach is used in Chapter 4 of [12]. Since we are mainly interested in how to approximate (P1) and (P2), we use the Galerkin method. In contrast to the approach presented in [12], where only the space V is discretized (semi-discretization), here we present the full approximation of both problems, introducing also suitable approximations of Y_1 and Y_2, respectively.

Our aim is now to develop a numerically available approximation model for the problem (P1) (we shall now consider only problem (P1), since (P2) can be treated in a similar way) and to prove that the solutions of this model tend to the solutions of the continuous problem. We shall show that the convergence is valid only for subsequences. This a consequence of the nonuniqueness of the solutions of the considered nonmonotone continuous and discrete problems. For simplicity we shall study in details only the approximation of the term $\int_{\Omega_0} \mathcal{X} \cdot v \, dx$, because the treatment of the bilinear form a and the linear form g is standard (see [2]).

Let $h \in (0,1)$ be a discretization parameter. Since we use a finite element method, h is related to the mesh size of partitions of $\overline{\Omega}$ and $\overline{\Omega}_0$ used for the constructions of finite-dimensional approximations V_h and Y_h of V and Y_1, respectively. We use the same discretization parameter h for both approximations.

First let us consider the space V. Let $\{V_h\}_{h\in(0,1)}$, $V_h \subset C(\overline{\Omega}; R^M)$, be a family of finite-dimensional subspaces of V. We denote by V_h' the dual space of V_h, $\langle \cdot, \cdot \rangle_{V_h}$ the corresponding duality pairing and $\| \cdot \|_{V_h}$ the norm of V_h induced by this one on V, i.e. $\|v_h\|_{V_h} = \|v_h\|_V$ for all $v_h \in V_h$. We shall assume that V_h, $h \in (0,1)$, are constructed in such a way that

$$\forall v \in V \; \exists \{v_h\}, \; v_h \in V_h : v_h \to v \text{ in } V \text{ as } h \to 0+ \, . \tag{8}$$

For example, V_h contains functions, components of which are piecewise polynomial over some triangulation of $\overline{\Omega}$.

The construction of Y_h is more involved. The crucial point is, how to approximate the integral $\int_{\Omega_0} \mathcal{X} \cdot v \, dx$, i.e. which quadrature formula will be used for the numerical integration. Let us fix a quadrature formula

$$\int_{\Omega_0} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{m_h} c_h^i f(x_h^i), \tag{9}$$

where c_h^i are weights and $x_h^i \in \overline{\Omega}_0$ are nodal points of the quadrature formula. We shall assume that (9) is exact for all constant functions so that $\sum_{i=1}^{m_h} c_h^i = m_N(\Omega_0)$, where m_N is the Lebesgue measure in R^N. Let us suppose that Ω_0 is such that the following construction is possible: We define a partition $\mathcal{T}_h$ of $\overline{\Omega}_h$, $\Omega_0 \subset \Omega_h$, such that it consists of subsets K_h^i of $\overline{\Omega}_h$ satisfying the following properties:

(i) $\overline{\Omega}_h = \cup_{i=1}^{m_h} K_h^i$;
(ii) $h \geq \max_{i=1,..,m_h}\{$*diameter of* $K_h^i\}$;
(iii) int $K_h^i \cap$ *int* $K_h^j = \emptyset\ \forall i \neq j$;
(iv) K_h^i *is closed, convex and has a nonempty interior for each* $i = 1, ..., m_h$;
(v) For each $i = 1, ..., m_h$ *there is exactly one point* $x_h^i \in$ *int* $K_h^i \cap \overline{\Omega}_0$;
(vi) $m_N($ *int* $K_h^i \cap \Omega_0) = c_h^i\ i = 1, ..., m_h$.

The finite element approximation Y_h of Y_1 will be defined as follows:

$$Y_h = \{\ f = (f_1, ..., f_M) \in L^\infty(\Omega_0; R^M) : \exists \tilde{f} = (\tilde{f}_1, ..., \tilde{f}_M) : \Omega_h \to R^M,$$
$$\tilde{f}_j|_{\text{int } K_h^i} \text{ is constant } i = 1, ..., m_h, j = 1, ..., M, f = \tilde{f}|_{\Omega_0}\},$$

i.e. Y_h contains all restrictions to Ω_0 of piecewise constant functions over $\mathcal{T}_h$. We shall identify Y_h with its dual Y_h'. The duality pairing and the norm in Y_h will be the same as in Y_1. We need also the space of the restrictions to $\overline{\Omega}_0$ of functions, the components of which are piecewise continuous functions over the partition $\mathcal{T}_h$:

$$X_h = \{\ f \in L^\infty(\Omega_0; R^M) : \exists \tilde{f} : \Omega_h \to R^M,$$
$$\tilde{f}_j|_{\text{int } K_h^i} \text{ is continuous } i = 1, ..., m_h, j = 1, ..., M, f = \tilde{f}|_{\Omega_0}\}.$$

Let P_h be a linear mapping from X_h to Y_h defined by means of

$$(P_h f)(x) = \sum_{i=1}^{m_h} f(x_h^i)(\mathcal{X}_{\text{int } K_h^i})(x), \quad x \in \Omega_0,$$

where $\mathcal{X}_{\text{int } K_h^i}$ is the characteristic function of int K_h^i. Then we have

$$\int_{\Omega_0} \sum_{j=1}^{M} (P_h f)_j(x)\,\mathrm{d}x = \sum_{i=1}^{m_h} \sum_{j=1}^{M} c_h^i f_j(x_h^i), \quad \forall f \in X_h, \tag{10}$$

using the definition of the mapping P_h and (vi).

The following consistency condition between V_h and Y_h is assumed to be satisfied in what follows:

$$\begin{aligned} &\text{for any } \{v_h\}, v_h \in V_h \text{ such that } v_h \rightharpoonup v \text{ (weakly) in } V \text{ as } h \to 0+ \\ &\Longrightarrow \text{there exists a subsequence } \{v_{h'}\} \subset \{v_h\} \text{ such that} \\ &P_{h'} v_{h'} \to v \text{ in } Y_1 \text{ as } h' \to 0+ . \end{aligned} \tag{11}$$

We also suppose that the norm of the linear operator P_h is bounded uniformly with respect to h:

$$\|P_h\|_{\mathcal{L}(V_h,Y_h)} \le C_4. \tag{12}$$

REMARK 3. In the case of the full discretization, considered here, the consistency condition (11) replaces the compactness property of the restriction mapping $r : V \to Y_1$, which is essential when proving the existence result for (P1) in [12]. In order to satisfy (11) and (12), the partitions used for the construction of V_h and Y_h has to be in a certain relation and also the nodes x_h^i have to be appropriately chosen. A particular case is presented in Section 5. For more details how to construct $\mathcal{T}_h$ and P_h, satisfying (i)–(vi) and (11),(12), respectively, we refer to [4],[5],[8],[10].

Now we are able to define the approximation of (P1) as follows:

$$\begin{cases} \text{find } u_h \in V_h \text{ and } \mathcal{X}_h(u_h) \in Y_h \text{ such that} \\ a(u_h, v_h) + \int_{\Omega_0} \mathcal{X}_h(x) \cdot (P_h v_h)(x)\,\mathrm{d}x = \langle g, v_h\rangle_V \quad \forall v_h \in V_h \\ \text{and } \mathcal{X}_h(x) \in \partial j((P_h u_h)(x)) \quad \text{a.e. in } \Omega_0 \end{cases} \tag{P1)$_h$}$$

or equivalently

$$\begin{cases} \text{find } u_h \in V_h \text{ and } \mathcal{X}_h(u_h) \in Y_h \text{ such that} \\ a(u_h, v_h) + \sum_{i=1}^{m_h} c_h^i \mathcal{X}_h(x_h^i) \cdot v_h(x_h^i) = \langle g, v_h\rangle_V \quad \forall v_h \in V_h \\ \text{and } \mathcal{X}_h(x_h^i) \in \partial j(u_h(x_h^i)) \quad i = 1, ..., m_h \end{cases} \tag{P1)$_h$}$$

making use of (10). The main result of the paper is that (P1)$_h$ is solvable and that its solutions tend on subsequences to solutions of (P1). More precisely we prove the following theorem:

THEOREM 1. *There exists at least one solution $(u_h, \mathcal{X}_h(u_h))$ of (P1)$_h$ for any $h \in (0,1)$ and we can find a subsequence $\{(u_{h'}, \mathcal{X}_{h'}(u_{h'}))\}$ of $\{(u_h, \mathcal{X}_h(u_h))\}$ such that $u_{h'}$ converges strongly to u in V and $\mathcal{X}_{h'}(u_{h'})$ converges weakly to $\mathcal{X}$ in Y_1.*

Moreover, $(u, \mathcal{X})$ is a solution of (P1).

REMARK 4. The counterpart of Theorem 1 holds also for the problem (P2) and it can be proved in a similar way. The construction of finite-dimensional approximation of Y_2 can be done as previously. First we fix a quadrature formula used for the approximation of the integral $\int_{\Gamma_0} \mathcal{X} \cdot v\,\mathrm{d}s$. Then we define a partition $\mathcal{T}_h$ of Γ_0, satisfying the properties (i)–(vi), finite-dimensional spaces X_h, Y_h and a linear operator P_h satisfying the conditions (11) and (12) (e.g. in [4],[5] there have been presented examples of operators P_h satisfying (11) and (12)) as for the problem (P1). Thus the approximation of (P2) can be formulated as follows:

$$\begin{cases} \text{find } u_h \in V_h \text{ and } \mathcal{X}_h(u_h) \in Y_h \text{ such that} \\ a(u_h, v_h) + \int_{\Gamma_0} \mathcal{X}_h(x) \cdot (P_h v_h)(x)\,\mathrm{d}s = \langle g, v_h\rangle_V \quad \forall v_h \in V_h \\ \text{and } \mathcal{X}_h(x) \in \partial j((P_h u_h)(x)) \quad \text{a.e. in } \Gamma_0. \end{cases} \tag{P2)$_h$}$$

3. Convergence Analysis for (P1)

In this section we shall prove Theorem 1. First we have to recall some definitions and results which will be used [3].

DEFINITION 2. Let T be a set-valued mapping from a Banach space X to a Banach space Y. Then T is said to be upper semicontinuous at $x \in X$ if the following property holds: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$T(x') \subset T(x) + \varepsilon B_Y \qquad \forall x' \in x + \delta B_X,$$

where B_X, B_Y are the unit balls in X, Y, respectively.

PROPOSITION 1. *Let $f : X \to R$ be locally Lipschitz near a point $x \in X$. Then it holds:*

(a) The function $f^\circ(x; z)$ is upper semicontinuous as a function of (x, z) in some neighbourhood $\mathcal{N}(x)$ of x, i.e.

$$\limsup_{y' \to y, z' \to z} f^\circ(y'; z') \leq f^\circ(y; z),$$

where $y, y' \in \mathcal{N}(x)$.

(b) $\partial f(x)$ is a nonempty, convex, weak $*$-compact subset of X' and $\|y\|_{X'} \leq K$ for every $y \in \partial f(x)$, where K is a Lipschitz constant of some neighbourhood $\mathcal{N}(x)$ of x.

(c) Let x_i and y_i be sequences in X and X' such that x_i converges to x, and that y is a cluster point of $y_i \in \partial f(x_i)$ in the weak $*$–topology. Then $y \in \partial f(x)$.

(d) If X is finite-dimensional, then ∂f is upper semicontinuous at x.

(e) For every z in X, one has

$$f^\circ(x; z) = \max\{\langle y, z\rangle_X : y \in \partial f(x)\}.$$

Let us also state a consequence of Kakutani fixed point theorem [1], which will be the main tool for proving Theorem 1.

THEOREM 2. *Let X be a finite-dimensional Banach space, T a coercive and upper semicontinuous set-valued mapping from X to X' such that Tx is a nonempty, bounded, closed, convex subset of X' for each $x \in X$. Then the range $R(T) = X'$.*

A set-valued operator $T : X \to X'$ is said to be coercive on a set $Z \subseteq X$ (with respect to 0) if

there exists a function $c : R_+ \to R$ with $\lim_{r\to\infty} c(r) = \infty$ such that

$$\langle y, x\rangle_X \geq c(\|x\|_X)\|x\|_X \qquad \text{for all } x \in Z \text{ and } y \in Tx.$$

Proof of Theorem 1. The proof of Theorem 1 consists of several steps.

(1°) First we show that (P1)$_h$ has a solution. We apply Theorem 2 to the operator equation

$$0 \in A_h u_h - g_h + T_h u_h, \tag{13}$$

where A_h is the operator from V_h to V_h' defined by $\langle A_h v_h, w_h \rangle_{V_h} = a(v_h, w_h)$ for all $v_h, w_h \in V_h$, g_h is an element of V_h' defined by $\langle g_h, v_h \rangle_{V_h} = \langle g, v_h \rangle_V$ for all $v_h \in V_h$ and T_h the set-valued operator from V_h to V_h' defined by

$$T_h v_h = \{ \ w_h \in V_h' : \exists z_h \in Y_h \text{ such that } w_h \equiv (P_h|_{V_h})^* z_h \\ \text{and } z_h(x) \in \partial j((P_h v_h)(x)) \text{ a.e. in } \Omega_0 \},$$

where $(P_h|_{V_h})^*$ means the transpose of $P_h|_{V_h}$.

From Proposition 1 (b) and the fact that $(P_h|_{V_h})^* : Y_h \to V_h'$ is a linear mapping it follows that $T_h v_h$ is a nonempty, bounded, closed, convex subset of V_h' for all $v_h \in V_h$. To see that T_h is upper semicontinuous we change the notation. Let $\{\varphi_h^j\}_{j=1}^{n_h}$, $n_h = \dim V_h$, be a basis of V_h. Let us make the identifications $V_h \equiv R^{n_h}$ and $Y_h \equiv [R^M]^{m_h}$, where m_h is the number of the nodal points of the quadrature formula (9). Then the mapping T_h can be identified with a mapping $\mathbf{T}$ from R^{n_h} to R^{n_h}:

$$\mathbf{Tv} = \{ \ \mathbf{w} \in R^{n_h} : \exists \mathbf{z} = (\mathbf{z}_1, ..., \mathbf{z}_{m_h}) \in [R^M]^{m_h} \text{ such that} \\ \mathbf{w} = (\mathcal{P})^* \mathbf{z} \text{ and } \mathbf{z}_i \in \partial j((\mathcal{P}\mathbf{v})_i) \quad i = 1, ..., m_h \},$$

where $\mathcal{P} = (\mathbf{P}_{ij})$ is an $m_h \times n_h$–matrix, the elements $\mathbf{P}_{ij}$ of which belong to R^M and are defined by

$$\mathbf{P}_{ij} = (P_h \varphi_h^j)(x_h^i), \quad i = 1, ..., m_h, j = 1, ..., n_h.$$

Thus, $\mathbf{T}$ is a composite function of a linear mapping $(\mathcal{P})^*$ and the set-valued mapping $\mathbf{v} \mapsto (\partial j((\mathcal{P}\mathbf{v})_1), ..., \partial j((\mathcal{P}\mathbf{v})_{m_h}))$, components of which are upper semicontinuous functions due to Proposition 1 (d). Then using the fact that $(\mathcal{P})^*$ is a linear continuous mapping from $[R^M]^{m_h}$ to R^{n_h}, the upper semicontinuity is still preserved as follows from Definition 2. This implies that also T_h is upper semicontinuous as a mapping from V_h to V_h'. Moreover, since the operator A_h is continuous, the mapping $A_h(\cdot) - g_h + T_h(\cdot)$ is upper semicontinuous from V_h to V_h'. On the other hand as the operator $A_h(\cdot) - g_h$ is single-valued then $A_h v_h - g_h + T_h v_h$ is a nonempty, bounded, closed, convex subset of V_h' for all $v_h \in V_h$.

Thus it remains only to show that the above mentioned operator is coercive. Let $v_h \in V_h$ and $w_h \in T_h v_h$. Then there exists $z_h \in Y_h$ such that $w_h = (P_h|_{V_h})^* z_h$ and $z_h(x) \in \partial j((P_h v_h)(x))$ a.e. in Ω_0. Using (1) and (12) we get

$$\langle w_h, v_h \rangle_{V_h} = \langle (P_h|_{V_h})^* z_h, v_h \rangle_{V_h} = \langle z_h, P_h v_h \rangle_{Y_h} \\ = - \int_{\Omega_0} z_h(x) \cdot (-(P_h v_h)(x)) \, \mathrm{d}x$$

$$\geq -\int_{\Omega_0} j^\circ((P_h v_h)(x); -(P_h v_h)(x))\,\mathrm{d}x \tag{14}$$
$$\geq -\int_{\Omega_0} (C_1 + C_2|(P_h v_h)(x)|^q)\,\mathrm{d}x \geq -\hat{C}_1 - \hat{C}_2\|P_h v_h\|^q_{Y_h}$$
$$\geq -\hat{C}_1 - \tilde{C}_2\|v_h\|^q_{V_h}.$$

From this and (4) we obtain that

$$\langle A_h v_h - g_h + (P_h|_{V_h})^* z_h, v_h\rangle_{V_h} = \langle A_h v_h - g_h, v_h\rangle_{V_h} + \langle z_h, P_h v_h\rangle_{Y_h}$$
$$\geq \alpha\|v_h\|^2_{V_h} - \|g_h\|_{V_h'}\|v_h\|_{V_h} - \hat{C}_1 - \tilde{C}_2\|v_h\|^q_{V_h}$$

implying the coerciveness of $A_h(\cdot) - g_h + T_h(\cdot)$. Thus the proof of this part is complete, because (13) is equivalent to (P1)$_h$.

(2°) Now, we prove that $\{u_h\}$, $\{\mathcal{X}_h\}$ are bounded in the corresponding function spaces. Let $(u_h, \mathcal{X}_h(u_h))$ be solutions of (P1)$_h$, $h \in (0, 1)$.

Taking the definition of (P1)$_h$ and substituting $v_h = u_h$ into it we get

$$a(u_h, u_h) = -\int_{\Omega_0} \mathcal{X}_h(x) \cdot (P_h u_h)(x)\,\mathrm{d}x + \langle g, u_h\rangle_V. \tag{15}$$

Because $\mathcal{X}_h(x) \in \partial j((P_h u_h)(x))$ a.e. in Ω, we can use (1) and (12) as in (14) to get

$$-\int_{\Omega_0} \mathcal{X}_h(x) \cdot (P_h u_h)(x)\,\mathrm{d}x \leq C_1' + C_2'\|u_h\|^q_V.$$

Substituting this into (15) and using (4) we see that

$$\alpha\|u_h\|^2_V \leq C_1' + C_2'\|u_h\|^q_V + \|g\|_{V'}\|u_h\|_V,$$

from which the boundedness of $\{u_h\}$ in V follows.

Due to (2) the Y_1-norm of $\mathcal{X}_h$ is bounded, i.e.,

$$\int_{\Omega_0} |\mathcal{X}_h(x)|^2\,\mathrm{d}x \leq C.$$

(3°) Now, we prove that cluster points of $\{u_h\}$ and $\{\mathcal{X}_h\}$ satisfy (P1).

Since V and Y_1 are Hilbert spaces, there exist subsequences $\{u_h\}$ and $\{\mathcal{X}_h\}$ (in the sequel we shall denote subsequences by the same symbol as the original ones) and elements $u \in V$ and $\mathcal{X} \in Y_1$ such that

$$u_h \rightharpoonup u \text{ in } V \text{ as } h \to 0+; \tag{16}$$
$$\mathcal{X}_h \rightharpoonup \mathcal{X} \text{ in } Y_1 \text{ as } h \to 0+. \tag{17}$$

First we show that u and $\mathcal{X}$ satisfy the equation

$$a(u, v) + \int_{\Omega_0} \mathcal{X} \cdot v\,\mathrm{d}x = \langle g, v\rangle_V \tag{18}$$

for all $v \in V$. Let $v \in V$ be given. Due to (8) there exists a sequence $\{v_h\}$, $v_h \in V_h$ such that $v_h \to v$ in V as $h \to 0+$. Then by applying (11), (16), (17) it is obvious that the following relations hold

$$a(u_h, v_h) \to a(u, v);$$
$$\int_{\Omega_0} \mathcal{X}_h \cdot P_h v_h \,\mathrm{d}x \to \int_{\Omega_0} \mathcal{X} \cdot v \,\mathrm{d}x;$$
$$\langle g, v_h\rangle_V \to \langle g, v\rangle_V, \quad \text{as } h \to 0+,$$

from which (18) follows.

It remains to show that

$$\mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega_0. \tag{19}$$

First we use (11) to get a subsequence of $\{u_h\}$ for which $\{P_h u_h\}$ converges strongly to u in Y_1. Then passing to a subsequence if necessary we may assume that the sequence $\{P_h u_h\}$ converges pointwisely to u a.e. in Ω_0.

Let $\phi \in L^\infty(\Omega_0; R^M)$ be given. Then

$$\int_{\Omega_0} \mathcal{X}(x) \cdot \phi(x) \,\mathrm{d}x = \lim_{h\to 0+} \int_{\Omega_0} \mathcal{X}_h(x) \cdot \phi(x) \,\mathrm{d}x$$
$$\le^{(1)} \limsup_{h\to 0+} \int_{\Omega_0} j^\circ((P_h u_h)(x); \phi(x)) \,\mathrm{d}x$$
$$\le^{(2)} \int_{\Omega_0} \limsup_{h\to 0+} j^\circ((P_h u_h); \phi(x)) \,\mathrm{d}x$$
$$\le^{(3)} \int_{\Omega_0} j^\circ(u(x); \phi(x)) \,\mathrm{d}x,$$

which implies (19), because ϕ was arbitrary. For the completeness let us justify the steps (1)–(3). The first step is a consequence of the inclusion $\mathcal{X}_h(x) \in \partial j((P_h u_h)(x))$ a.e. in Ω_0 and the definition of the generalized gradient. The second step is due to the Fatou's lemma and the following arguments: Because of (2) any element η from $\partial j((P_h u_h))$ the inequality $|\eta| \le C_3(1 + |(P_h u_h)(x)|)$ holds, we can deduce from Proposition 1 (e) that $j^\circ((P_h u_h)(x); \phi(x)) \le C_3(1 + |(P_h u_h)(x)|)|\phi(x)|$. Moreover, we know that $P_h u_h$ converges strongly in Y_1 and pointwisely a.e. in Ω_0 to u. Thus we can apply the Fatou's lemma to get the inequality (2). The third step is due to the upper semicontinuity of the generalized directional derivative (see Proposition 1 (a)) and the pointwise convergence of $P_h u_h$ to u.

(4°) It remains to prove the strong convergence of $\{u_h\}$ to u. Because of (8), there exists $\{\bar{u}_h\}$, $\bar{u}_h \in V_h$ such that $\bar{u}_h \to u$ in V as $h \to 0+$. Using (4) and the fact that u_h is a solution of (P1)$_h$ we see that

$$\alpha\|u_h - \bar{u}_h\|^2 \le a(u_h - \bar{u}_h, u_h - \bar{u}_h) = \langle g, u_h - \bar{u}_h\rangle_V$$
$$- \int_{\Omega_0} \mathcal{X}_h(x) \cdot (P_h(u_h - \bar{u}_h))(x) \,\mathrm{d}x - a(\bar{u}_h, u_h - \bar{u}_h). \tag{20}$$

Passing to a subsequence if necessary, the right handside of (20) tends to zero as $h \to 0+$ due to (11). Using triangle inequality wet get the strong convergence of the sequence $\{u_h\}$ to u in V.

4. Constrained Vector–Valued Hemivariational Inequalities

In this section we shall consider slightly modified problems, namely the so called *constrained vector-valued hemivariational inequalities*. The mathematical formulation of these problems is similar to the formulation of the unconstrained hemivariational inequalities (P1) and (P2) except we have now a *nonempty, closed, convex subset* K of V in which we look for solutions, i.e.

$$\begin{cases} \text{find } u \in K \text{ and } \mathcal{X}(u) \in Y_1 \text{ such that} \\ a(u, v-u) + \int_{\Omega_0} \mathcal{X} \cdot (v-u)\,\mathrm{d}x \geq \langle g, v-u \rangle_V \quad \forall v \in K \\ \text{and } \mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega_0 \end{cases} \tag{CP1}$$

or

$$\begin{cases} \text{find } u \in K \text{ and } \mathcal{X}(u) \in Y_2 \text{ such that} \\ a(u, v-u) + \int_{\Gamma_0} \mathcal{X} \cdot (v-u)\,\mathrm{d}s \geq \langle g, v-u \rangle_V \quad \forall v \in K \\ \text{and } \mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Gamma_0. \end{cases} \tag{CP2}$$

REMARK 5. It would be possible to consider problems (P1),(P2) from Section 3 as special cases of (CP1),(CP2), respectively. For the convenience of readers we prefer to present results separately.

As previously these can be transformed into the equivalent operator inclusions:

$$\text{find } u \in V \text{ such that } 0 \in Au - g + T_1 u + \partial I_K(u); \tag{CP1}$$

or

$$\text{find } u \in V \text{ such that } 0 \in Au - g + T_2 u + \partial I_K(u), \tag{CP2}$$

where $\partial I_K(u)$ is the generalized gradient of the indicator function I_K of K. Since K is nonempty, closed and convex, ∂I_K coincides with the subdifferential of the convex function I_K. Moreover, ∂I_K is a maximal monotone operator (see [18]).

Let us start to study the approximation of these problems. We shall formulate exactly the same assumptions as for problems (P1) and (P2) (as before we shall consider (CP1) only because the treatment of (CP2) is similar). Instead of V_h we have to construct the approximation of the convex set K, but this can be done in a standard way (see [4],[5],[6]):

Let $\{K_h\}$ be a family of nonempty, closed convex subsets of V_h (not necessarily $K_h \subset K$) satisfying

$$\forall v \in K\ \exists \{v_h\},\ v_h \in K_h : v_h \to v \text{ in } V \text{ as } h \to 0+; \tag{21}$$

$$\{v_h\},\ v_h \in K_h : v_h \rightharpoonup v \text{ in } V \text{ as } h \to 0+ \Longrightarrow v \in K. \tag{22}$$

The approximation of (CP1) reads as follows:

$$\begin{cases} \text{find } u_h \in K_h \text{ and } \mathcal{X}_h(u_h) \in Y_h \text{ such that} \\ a(u_h, v_h - u_h) + \int_{\Omega_0} \mathcal{X}_h \cdot (P_h v_h - P_h u_h)\,\mathrm{d}x \\ \geq \langle g, v_h - u_h\rangle_V \quad \forall v_h \in K_h \\ \text{and } \mathcal{X}_h(x) \in \partial j((P_h u_h)(x)) \quad \text{a.e. in } \Omega_0. \end{cases} \qquad (\text{CP1})_h$$

Next we shall prove:

THEOREM 3. *There exists at least one solution $(u_h, \mathcal{X}_h(u_h))$ of (CP1)$_h$ for any $h \in (0, 1)$. We can find a subsequence $\{(u_{h'}, \mathcal{X}_{h'}(u_{h'}))\}$ of $\{(u_h, \mathcal{X}_h(u_h))\}$ such that $u_{h'}$ converges strongly to u in V and $\mathcal{X}_{h'}(u_{h'})$ converges weakly to $\mathcal{X}$ in Y_1.*

Moreover, $(u, \mathcal{X})$ is a solution of (CP1).

REMARK 6. The counterpart of Theorem 3 holds for the problem (CP2), as well.

The solvability proof of (CP1)$_h$ is based on the following existence result (see [1]) for the upper semicontinuous set-valued operators:

THEOREM 4. *Let K be a closed, convex subset of a reflexive Banach space X such that $0 \in K$, and let F be a finite-dimensional subspace of X. Let T be a set-valued mapping from $K \cap F$ into X' such that for each $x \in K \cap F$, Tx is a nonempty, bounded, closed and convex subset of X'. Suppose that T is upper semicontinuous from $K \cap F$ to the weak topology of X', and that T is coercive on $K \cap F$. Then there exists $x_0 \in K \cap F$ and $y_0 \in Tx_0$ such that for all $x \in K \cap F$,*

$$\langle y_0, x - x_0\rangle_X \geq 0. \tag{23}$$

Let us assume that X is a finite-dimensional Banach space. Then the following corollary of Theorem 4 holds:

COROLLARY 1. *Let K be a closed, convex subset of the finite-dimensional Banach space X such that $0 \in K$. Let T be a set-valued mapping from K into X' such that for each $x \in K$, Tx is a nonempty, bounded, closed and convex subset of X'. Suppose that T is upper semicontinuous from K to X', and that T is coercive on K. Then there exists $x_0 \in K$ and $y_0 \in Tx_0$ such that for all $x \in K$,*

$$\langle y_0, x - x_0\rangle_X \geq 0.$$

REMARK 7. Corollary 1 holds true also in the case when the assumptions that $0 \in K$ and T is coercive on K are replaced by the following ones: There is an element $\bar{x} \in K$ such that T is coercive with respect to $\bar{x}$ on K, i.e. there exists a function $c : R_+ \to R$ with $\lim_{r\to\infty} c(r) = \infty$ such that for all $x \in K$ and $y \in T(x)$ it holds: $\langle y, x - \bar{x}\rangle_X \geq c(\|x\|_X)\|x\|_X$. This can be shown by applying Corollary 1 to the mapping $T_{\bar{x}}(x) = T(x + \bar{x})$ defined on the set $K_{\bar{x}} = K - \bar{x}$.

Proof of Theorem 3. The proof will be done in several steps. Let us first fix some $u_0 \in K$ and a sequence $\{u_0^h\}$, $u_0^h \in K_h$ such that $u_0^h \to u_0$ in V, which exists due to (21).

(1°) First we prove the existence of a solution of (CP1)$_h$. The idea is to apply Corollary 1 and Remark 7 to the set-valued mapping $A_h(\cdot) - g_h + T_h(\cdot)$ introduced in the proof of Theorem 1. We have already shown that this mapping is upper semicontinuous and $A_h v_h - g_h + T_h v_h$ is a nonempty, bounded, closed and convex subset of V_h' for all $v_h \in V_h$. It remains to show that $A_h(\cdot) - g_h + T_h(\cdot)$ is coercive on K_h with respect to u_0^h. Let $v_h \in V_h$ and $w_h \in T_h v_h$. Then there exists $z_h \in Y_h$ such that $w_h = (P_h|_{V_h})^* z_h$ and $z_h(x) \in \partial j((P_h v_h)(x))$ a.e. in Ω_0.

Using (1), (2) and (12) we get

$$
\begin{aligned}
&\langle w_h, v_h - u_0^h\rangle_{V_h} = \langle (P_h|_{V_h})^* z_h, v_h - u_0^h\rangle_{V_h} = \langle z_h, P_h v_h - P_h u_0^h\rangle_{Y_h} \\
&= -\int_{\Omega_0} z_h(x)\cdot(-(P_h v_h)(x))\,\mathrm{d}x - \int_{\Omega_0} z_h(x)\cdot(P_h u_0^h)(x)\,\mathrm{d}x \\
&\geq -\int_{\Omega_0} j^\circ((P_h v_h)(x); -(P_h v_h)(x))\,\mathrm{d}x \\
&\quad -\int_{\Omega_0} C_3(1 + |P_h v_h(x)|)|(P_h u_0^h)(x)|\,\mathrm{d}x \\
&\geq -\int_{\Omega_0} (C_1 + C_2|(P_h v_h)(x)|^q)\,\mathrm{d}x \qquad\qquad (24)\\
&\quad -\Big(\int_{\Omega_0} (C_3(1 + |P_h v_h(x)|))^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{\Omega_0} |P_h u_0^h(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\geq -\hat{C}_1 - \hat{C}_2\|P_h v_h\|_{Y_h}^q - C(1 + \|P_h v_h\|_{Y_h})\|P_h u_0^h\|_{Y_h} \\
&\geq -\hat{C}_1 - \tilde{C}_2\|v_h\|_{V_h}^q - \hat{C}(1 + \|v_h\|_{V_h})\|u_0^h\|_{V_h}.
\end{aligned}
$$

From (3), (4) and (24) we obtain that

$$
\begin{aligned}
&\langle A_h v_h - g_h + (P_h|_{V_h})^* z_h, v_h - u_0^h\rangle_{V_h} \\
&= \langle A_h v_h - g_h, v_h - u_0^h\rangle_{V_h} + \langle z_h, P_h v_h - P_h u_0^h\rangle_{Y_h} \\
&\geq \alpha\|v_h\|_{V_h}^2 - \|g_h\|_{V_h'}(\|v_h\|_{V_h} + \|u_0^h\|_{V_h}) - m\|v_h\|_{V_h}\|u_0^h\|_{V_h} \\
&\quad -\hat{C}_1 - \tilde{C}_2\|v_h\|_{V_h}^q - \hat{C}(1 + \|v_h\|_{V_h})\|u_0^h\|_{V_h}
\end{aligned}
$$

implying the coerciveness of $A_h(\cdot) - g_h + T_h(\cdot)$ on K_h with respect to u_0^h. Then the existence a solution of (CP1)$_h$ follows from Corollary 1 and Remark 7. The proof of this part is now complete.

(2°) Next we prove that $\{u_h\}$ and $\{\mathcal{X}_h\}$ are bounded in V and Y_1, respectively. Let $(u_h, \mathcal{X}_h(u_h))$ be solutions of (CP1)$_h$, $h \in (0,1)$. Substituting $v_h = u_0^h$ into the definition of (CP1)$_h$ we obtain

$$
\begin{aligned}
a(u_h, u_h) \leq a(u_h, u_0^h) + \int_{\Omega_0} \mathcal{X}_h(x)\cdot((P_h u_0^h)(x) \\
-(P_h u_h)(x))\,\mathrm{d}x + \langle g, u_0^h - u_h\rangle_V. \qquad (25)
\end{aligned}
$$

Taking into account that $\mathcal{X}_h(x) \in \partial j((P_h u_h)(x))$ a.e. in Ω_0, we can deduce as in (24) that

$$-\int_{\Omega_0} \mathcal{X}_h(x) \cdot ((P_h u_h)(x) - (P_h u_0^h h)(x)\,\mathrm{d}x$$
$$\leq C_1' + C_2' \|u_h\|_V^q + C'(1 + \|u_h\|_V)\|u_0^h\|_V.$$

Substituting this to (25) and using (4) we see that

$$\alpha \|u_h\|_V^2 \leq m\|u_h\|_V \|u_0^h\|_V + C_1' + C_2'\|u_h\|_V^q$$
$$+C'(1 + \|u_h\|_V)\|u_0^h\|_V + \|g\|_{V'}(\|u_0^h\|_V + \|u_h\|_V),$$

which implies the boundedness of $\{u_h\}$ in V. The boundedness of $\{\mathcal{X}_h\}$ in Y_1 can be shown in a similar way as in the proof of Theorem 1.

(3°) The fact that cluster points of $\{u_h\}$ and $\{\mathcal{X}_h\}$ satisfy (CP1) can be shown exactly in the same way as previously. The only thing is to note that if we have a sequence $\{u_h\}$ converging weakly to u in V, a sequence $\{v_h\}$ converging strongly to v in V and a sequence $\{\mathcal{X}_h\}$ converging weakly to $\mathcal{X}$ in Y_1, we have (passing to subsequences if necessary):

$$\limsup_{h \to 0+} a(u_h, v_h - u_h) \leq a(u, v - u);$$
$$\int_{\Omega_0} \mathcal{X}_h \cdot (P_h v_h - P_h u_h)\,\mathrm{d}x \to \int_{\Omega_0} \mathcal{X} \cdot (v - u)\,\mathrm{d}x;$$
$$\langle g, v_h - u_h \rangle_V \to \langle g, v - u \rangle_V,$$

as $h \to 0+$.

(4°) Also the strong convergence of $\{u_h\}$ can be proved in a similar way as in the proof of Theorem 1. The only modification is that we use (21) (not (8)) to get a sequence $\{\bar{u}_h\}$, $\bar{u}_h \in K_h$ such that $\bar{u}_h \to u$ in V as $h \to 0+$.

5. Applications

Here we present one example, the approximation of which is based on results of the previous sections.

Nonmonotone skin friction in plane elasticity (see [12]): Let us assume a plane elastic body, represented by a polygonal domain $\Omega \subset R^2$ with the Lipschitz boundary Γ. The equilibrium state of Ω is described by *the system of equilibrium* equations:

$$\sigma_{ij,j} + F_i = 0 \quad \text{in } \Omega, \quad i = 1, 2, \tag{26}$$

where the stress tensor $\sigma = (\sigma_{ij})_{i,j=1}^2$ is related to the linearized strain tensor $\varepsilon = (\varepsilon_{ij})_{i,j=1}^2$ by means of a linear Hooke's law

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \text{where } \varepsilon_{kl}(u) = \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}) \tag{27}$$

and c_{ijkl} are elasticity coefficients, satisfying the usual symmetry and ellipticity conditions in Ω. For simplicity let us assume that displacements $u = (u_1, u_2)$ are equal to zero on Γ.

In order to describe the skin effects, we split F into two parts: $F = \overline{F} + \overline{\overline{F}}$. The part $\overline{\overline{F}}$ is given *a priori* and represents an external loading on Ω. The part $\overline{F}$ (possibly multivalued) is induced by skin effects, arising on Ω_0, $\Omega_0 \subset\subset \Omega$. Therefore $\overline{F} = 0$ on $\Omega \setminus \Omega_0$. Let us consider that the multivalued constitutive (reaction-displacement) law is expressed in the form

$$-\overline{F}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega_0, \tag{28}$$

where $j : R^2 \to R$ is a locally Lipschitz continuous function, satisfying (1) and (2). The weak formulation of our problem, which is described by (26)- (28) reads as follows:

$$\begin{cases} \text{find } u \in (H_0^1(\Omega))^2 \text{ and } \mathcal{X} \in (L^2(\Omega_0))^2 \text{ such that} \\ (\sigma(u), \varepsilon(v))_{0,\Omega} + (\mathcal{X}, v)_{0,\Omega_0} = (\overline{\overline{F}}, v)_{0,\Omega} \quad \forall v \in (H_0^1(\Omega))^2 \\ \text{and } \mathcal{X}(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega_0. \end{cases} \tag{29}$$

For the approximation of (29) we shall use the finite element technique. Let $\{\mathcal{T}_h\}$, $h \to 0+$ be a regular family of triangulations of $\overline{\Omega}$. With any $\mathcal{T}_h$ the space of piecewise linear functions will be associated:

$$V_h = \{v_h \in (C(\overline{\Omega}))^2 \mid v_h|_T \in (P_1(T))^2 \; \forall T \in \mathcal{T}_h, \; v_h = 0 \text{ on } \Gamma\}.$$

On any triangle $T \in \mathcal{T}_h$ we shall consider the following quadrature formula:

$$\int_T f(x)\,\mathrm{d}x \approx \frac{1}{3} m_2(T)(f(M^{1T}) + f(M^{2T}) + f(M^{3T})), \tag{30}$$

where $m_2(T)$ is the area of T and M^{jT}, $j = 1, 2, 3$ are the midpoints of the edges of T. For simplicity let us assume that also Ω_0 is a polygonal domain such that $\overline{\Omega}_0 = \cup_{i \in I} \overline{T}_i$, where $I = \{i \mid T_i \in \mathcal{T}_h \text{ and int } T_i \cap \Omega_0 \neq \emptyset\}$, i.e. Ω_0 is a union of triangles, belonging to the original triangulation $\mathcal{T}_h$, the interior of which has a nonempty intersection with Ω_0. Since Ω_0 is polygonal, the integration formula (10) over Ω_0 is given by taking a sum of (30) over all $T \in I$:

$$\int_{\Omega_0} f(x)\,\mathrm{d}x \approx \sum_{T \in I} \frac{1}{3} m_2(T)(f(M^{1T}) + f(M^{2T}) + f(M^{3T})). \tag{31}$$

Rearranging terms in (31), we finally obtain

$$\int_{\Omega_0} f(x)\,\mathrm{d}x \approx \sum_i c_h^i f(x_h^i),$$

where $x_h^i = M^{jT}$ for some $j = 1, 2, 3$ and $T \in I$, while $c_h^i = \frac{1}{3} m_2(T)$ or $c_h^i = \frac{1}{3}(m_2(T) + m_2(T'))$ if the corresponding x_h^i is on $\partial\Omega_0$ or in the interior

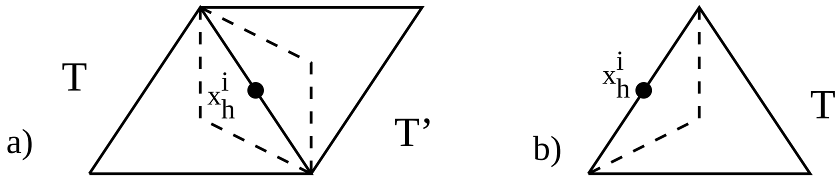

Figure 1. a) $c_h^i = \frac{1}{3}(m_2(T) + m_2(T'))$. b) $c_h^i = \frac{1}{3}m_2(T)$.

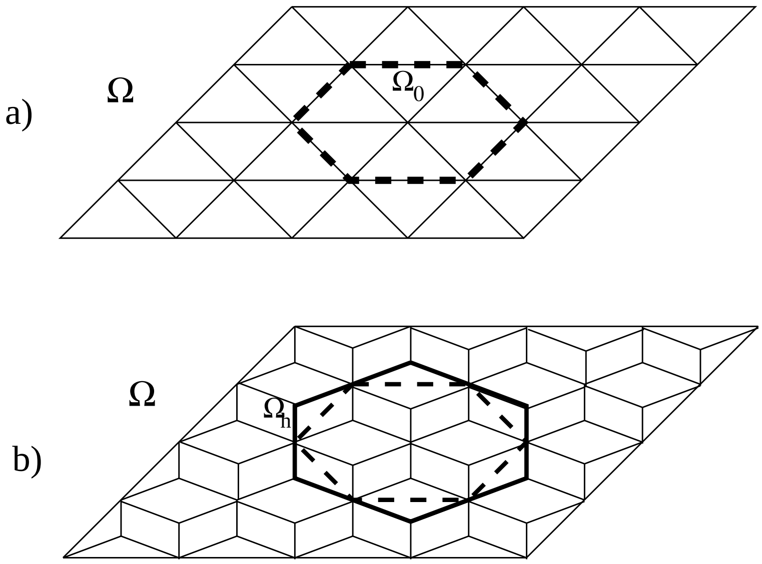

Figure 2. Partitions $\mathcal{T}_h$ and $\mathcal{T}_h^1$.

of Ω_0, respectively. Here T and T' are 2 adjacent triangles from I with x_h^i as the common midpoint (see Figure 1).

With any triangulation $\mathcal{T}_h$ we associate another partition $\mathcal{T}_h^1$ of $\overline{\Omega}$, which can be constructed as follows:

(i) it consists of quadrilaterals, nodes of which are vertices of 2 adjacent triangles $T', T \in \mathcal{T}_h$ and their centre of gravities.

(ii) triangles in the case, when one edge of $T \in \mathcal{T}_h$ is on the boundary of Ω.

So we can write

$$\overline{\Omega} = \cup_{T \in \mathcal{T}_h} T = \cup_{Q \in \mathcal{T}_h^1} Q,$$

where Q's are elements of the new partition $\mathcal{T}_h^1$ (see Figure 2a the partition $\mathcal{T}_h$ and Figure 2b the partition $\mathcal{T}_h^1$).

Denote by $\overline{\Omega}_h = \cup Q$, where the union is taken over all Q's, the interior of which has a nonempty intersection with Ω_0. These Q's will play the role of the subsets K_h^i introduced in Section 2, and used when defining the space Y_h. Since by assumption dist $(\partial\Omega_0, \partial\Omega) > 0$, each K_h^i contains one integration point x_h^i in its interior. It is readily seen that (i)–(vi) of Section 2 are satisfied. It is also easy to show (see [8], [10]) that in this case conditions (11) and (12) are satisfied. Since at the same time, condition (8) is satisfied for our system $\{V_h\}$, the corresponding discrete problems solved on $V_h \times Y_h$ are close on subsequences to the continuous one.

REMARK 8. For the completeness let us sketch shortly how one can prove (11).

$$\begin{aligned}
&\|P_h v_h - v_h\|^2_{Y_1} \\
&= \sum_{Q \in \mathcal{T}_h^1} \sum_{i=1,2} \int_{Q\cap\Omega_0} |(P_h v_h)_i(x) - (v_h)_i(x)|^2 \, \mathrm{d}x \\
&\leq \sum_{Q \in \mathcal{T}_h^1} \sum_{i=1,2} \int_{Q\cap\Omega_0} h^2 |\nabla (v_h)_i(x)|^2 \, \mathrm{d}x \\
&\leq h^2 \|v_h\|^2_{H^1(\Omega_0;R^2)}.
\end{aligned}$$

Then, because of $v_h \rightharpoonup v$ in V and the triangle inequaity we see immediatelty that $\{P_h v_h\}$ converges strongly to v in Y_1 as $h \to 0+$.

REMARK 9. There is an alternative way how to construct Y_h. The sets K_h^i used for the definition of Y_h are formed now by polygons, constructed as follows: Let N_i be a node of $\mathcal{T}_h$. We define K_h^i as a polygon bounded by segments, joining centroids of all $T \in \mathcal{T}_h$, having N_i as a common vertex, to the midpoint of the edges, containing N_i. Such construction is described in [4],[5].

REMARK 10. Let us shortly describe the numerical realization of the discretized nonmonotone skin friction problem. Using the notations introduced in the proof of Theorem 1 we can rewrite it into the matrix form as follows (as the discretization parameter h is fixed, we skip it):

$$\begin{cases}
\text{find } \mathbf{u} = (u_1, ..., u_n) \in R^n \text{ and } \mathbf{s} = (\mathbf{s}_1, ..., \mathbf{s}_m) \in [R^M]^m \\
\text{such that} \quad (\mathbf{A}\mathbf{u}, \mathbf{v})_{R^n} + (\mathbf{s}, \mathcal{P}\mathbf{v})_{[R^M]^m} = (\mathbf{g}, \mathbf{v})_{R^n} \quad \forall \mathbf{v} \in R^n \\
\text{and } \mathbf{s}_i \in c_i \partial j((\mathcal{P}\mathbf{u})_i) \quad i = 1, ..., m,
\end{cases} \tag{P}$$

where $\mathbf{A} = (\langle A_h \varphi^i, \varphi^j \rangle_{V_h})_{i,j=1}^n$, $\mathbf{g} = (\langle g_h, \varphi^j \rangle_{V_h})_{j=1}^n \in R^n$ and $(\cdot,\cdot)_{R^n}$, $(\cdot,\cdot)_{[R^M]^m}$ denote the scalar products of R^n and $[R^M]^m$, respectively. Now due to the symmetry of the elasticity coefficients, the matrix $\mathbf{A}$ is also symmetric. Therefore one possibility how to solve (P) is tranform it to a problem of finding local minima or more generally substationary points of the corresponding potential function $L : R^n \to R$, i.e. points $\mathbf{w} \in R^n$ such that $0 \in \partial L(\mathbf{w})$, where L is defined by

$$L(\mathbf{v}) = \frac{1}{2}(\mathbf{A}\mathbf{v}, \mathbf{v})_{R^n} - (\mathbf{g}, \mathbf{v})_{R^n} + \Psi(\mathbf{v}),$$

where

$$\Psi(\mathbf{v}) = \sum_{i=1}^{m} c_i j((\mathcal{P}\mathbf{v})_i).$$

It is possible to show that all substationary points, especially all local minima, of L are also solutions of the problem (P) (see [8],[10]). To find the local minima we can use optimization methods for nonsmooth, nonconvex functions (see [11]), as the function L is nonsmooth and nonconvex in general. The other possibility how to solve (P) is to use an iterative method where the nonmonotone problem (P) is approximated by a sequence of monotone subproblems which can be solved more effectively (see [17]).

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